MATH 207(c)–Analysis IV

1. Compute the value of the following integrals (*Hint: You might want to use Cauchy's integral formula to get the answer without long computations!*):

(a) $\displaystyle\int_\gamma \frac{e^{2z}}{z}\,dz$ with $\gamma = \{z : |z| = 2\}$ oriented counter-clockwise.

(b) $\displaystyle\int_\gamma \frac{z^3 + 2z^2 + 2}{z - 2i}\,dz$ with $\gamma = \{z : |z - 2i| = 1\}$ oriented clockwise.

(c) $\displaystyle\int_\gamma \frac{\sin(3z + \frac{\pi}{4})}{(z - \pi)^2}\,dz$ with $\gamma = \{z : |z - \pi| = 3\}$ oriented counter-clockwise.

2. Similarly, compute the following integrals:

(a) $\displaystyle\int_\gamma \frac{e^{2z}}{(z-1)(z^2+4)}\,dz$ where γ is the bounday of the rectangle

$$\mathcal{R} = \{z : -2 \leqslant \mathrm{Re}(z) \leqslant 2,\ -1 \leqslant \mathrm{Im}(z) \leqslant 1\}$$

oriented clockwise.

(b) $\displaystyle\int_\gamma \frac{\sin(z^2)}{\cos(z)}\,dz$ with $\gamma = \{z : |z - i| = 1\}$ oriented counter-clockwise. *(Hint: You might want to first compute the zeroes of the* $\cos(\cdot)$ *function by examining its real and imaginary parts.)*

3. Let γ be a *simple, closed and counter-clockwise oriented* regular curve in $\mathbb{C}$. What are the possible values the following integral can take (depending on the exact form of γ):

$$\int_\gamma \frac{\cosh(z^2+1)}{(z-2)^3}\,dz.$$

4. For

$$f(z) = \frac{e^{iz}}{(z-i)^2},$$

compute the integral $\displaystyle\int_\gamma f(z)\,dz$ in the following cases:

(a) $\gamma = \{z : |z| = 2\}$ oriented clockwise.

(b) γ is the boundary of the rectangle $\{z : |\mathrm{Re}(z)| \leqslant 4, |\mathrm{Im}(z)| \leqslant \frac{1}{2}\}$ oriented counter-clockwise.

(c) For some $R > 1$, γ is the closed curve formed by the union $\gamma_1 \cup \gamma_2$, where $\gamma_1(t) = t$ for $-R \leqslant t \leqslant R$ and $\gamma_2(s) = Re^{is}$ for $s \in [0, \pi]$ (draw a picture to visualise the curve in the comlex plane).
Bonus question: In the limit $R \to +\infty$, show that the integral $\int_{\gamma_2} f(z)\, dz$ goes to 0. Based on that, can you compute the value of $\int_{\mathbb{R}} f(z)\, dz$?

5. Complex integrals have historically proved to be a valuable tool in calculating complicated integral expressions, even in cases where the starting integrals do not seem to involve complex numbers at all. To illustrate this, we will use the techniques of complex integration to calculate

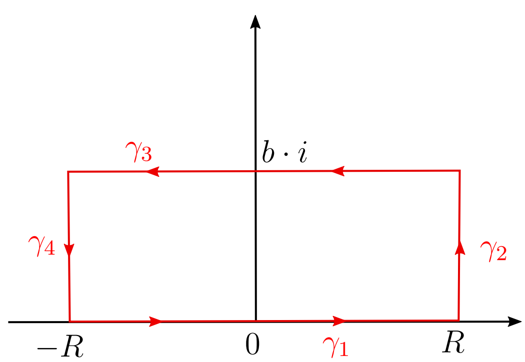
$$\int_{-\infty}^{+\infty} e^{-x^2} \cos(2bx)\, dx = \sqrt{\pi} e^{-b^2}, \tag{1}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} \sin(2bx)\, dx = 0,$$

where $b > 0$ is any (real) given constant.

(a) Show that the function $f(z) = e^{-z^2}$ is entire.

(b) For any $R > 0$, consider the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ considered below:



Show that $\displaystyle\int_\gamma f(z)\, dz = 0$.

(c) Show that, as $R \to +\infty$, $\int_{\gamma_2} f(z)\, dz$, $\int_{\gamma_4} f(z)\, dz \to 0$.

(d) Using the fact that $\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$, deduce from the above that (1) holds.

6. Let $\mathcal{D} \subset \mathbb{C}$ be an open set and $f : \mathcal{D} \to \mathbb{C}$ be a continuous function. Assume that there exists a holomorphic function $F : \mathcal{D} \to \mathbb{C}$ which is an antiderivative of f; this means that $F'(z) = f(z)$. Show that, for any regular curve $\gamma : [a, b] \to \mathcal{D}$,

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

(Hint: You will need to use the chain rule to compute $\frac{d}{dt}\Big(F(\gamma(t))\Big)$.)
In the case when $\mathcal{D} = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$, show that no antiderivative of f exists on $\mathcal{D}$. How is this consistent with what we know about $\log(z)$?

Remark: In the case when $\mathcal{D}$ is simply connected, the above formula can be used to show that any holomorphic $f : \mathcal{D} \to \mathbb{C}$ has an antiderivative.

Solutions

1. Let us recall Cauchy's integral formula: $\displaystyle\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$ where γ is a simple, closed and positively oriented (i.e. counter clockwise) path inside an open and simply connected set $\mathcal{D} \subseteq \mathbb{C}$, $z_0 \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbb{C}$ is holomorphic. Recall that, in practice, it suffices for f to be holomorphic in the interior of γ.

(a)

$$\int_\gamma \frac{e^{2z}}{z}\,dz = \int_\gamma \frac{e^{2z}}{z-0}\,dz = 2\pi i e^{2\cdot 0} = 2\pi i$$

The expression is valid since the function $f(z) = e^{2z}$ is holomorphic in $\mathcal{D} = \mathrm{Int}(\gamma) = \mathcal{C}(0,2)$

(b)

$$\int_{\gamma_-} \frac{z^3+2z^2+2}{z-2i}\,dz = -\int_{\gamma_+} \frac{z^3+2z^2+2}{z-2i}\,dz \overset{f(z)=z^3+2z^2+2}{=} -2\pi i f(2i)$$

$$= -2\pi i(-8i-8+2) = -16\pi + 12\pi i$$

Since we originally want to compute this integral along the negatively (i.e. clockwise) oriented closed path γ_-, we use the fact that when we switch the orientation of a curve, the corresponding integral changes sign. The plus (minus) sign refers to a counter-clockwise (clockwise) orientation.

(c)

$$\int_\gamma \frac{\sin(3z+\frac{\pi}{4})}{(z-\pi)^2}\,dz \overset{f(z)=\sin(3z+\frac{\pi}{4})}{=} \frac{2\pi i}{1!} f'(\pi) = 2\pi i \left.\frac{d\sin(3z+\pi/4)}{dz}\right|_{z=\pi}$$

$$= 6\pi i \cos(\pi+\pi/4) = -3\sqrt{2}\pi i.$$

2. Similarly to exercise 1, we make an extensive use of the Cauchy integral formula to avoid difficult computations when possible:

(a)

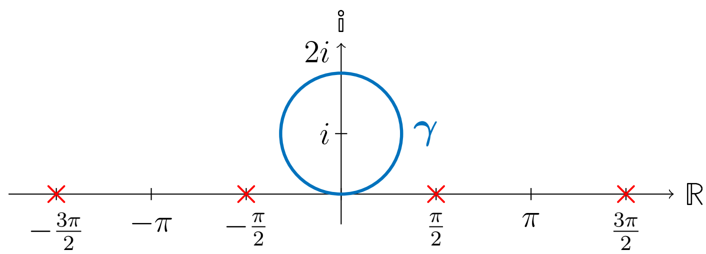
$$\int_{\gamma_-} \frac{e^{2z}}{(z-1)(z^2+4)}\,dz = -\int_{\gamma_+} \frac{e^{2z}}{(z-1)(z^2+4)}\,dz = -2\pi i f(1) = -2\pi i \cdot \frac{e^{2\cdot 1}}{1^2+4} = -\frac{e^2}{5}$$

where the function $f(z) = e^{2z}/(z^2+4)$ is holomorphic over the rectangular domain defined as $\mathcal{R} = \{z :\ -2 \leqslant \mathrm{Re}(z) \leqslant 2,\ -1 \leqslant \mathrm{Im}(z) \leqslant 1\}$. The sign change reflects the clockwise integration direction.

(b) Let us first look at the zeros of the complex cosine function:

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2} = \frac{e^{-y}e^{ix}+e^{y}e^{-ix}}{2} \\ &= \frac{e^{-y}}{2}(\cos(x)+i\sin(x)) + \frac{e^{y}}{2}(\cos(x)-i\sin(x)) \\ &= \cos(x)\left(\frac{e^{y}+e^{-y}}{2}\right) - i\sin(x)\left(\frac{e^{y}-e^{-y}}{2}\right) \\ &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \quad \text{with} \quad \{x,y\}\in\mathbb{R}\end{aligned}$$

we see that the real part is only zero when $x_k = \pi/2 + k\pi$ ($\cosh(y)$ is never equal to zero), and implies for the imaginary part that $y = 0$. We conclude that the complex cosine has the same zeros as the real function. Returning to the integral, the path γ is a circle centered at i of radius 1. We notice that all the singular points of $f(z) = \sin(z^2)/\cos(z)$, i.e. the zeros of $\cos(z)$, lie outside the integration domain defined by γ. The distance from the center of the circle to one of the closest zeros $d = \sqrt{1^2 + (\pi/2)^2} \simeq 1.58$ is larger than the radius of the circle.



The zeros of $\cos(z)$ are indicated by the red crosses.

Using Cauchy's theorem, the integral is simply $\displaystyle\int_\gamma \frac{\sin(z^2)}{\cos(z)}\,dz = 0$.

3. We can distinguish three different case scenarios, all depending on the location of the zero $z_0 = 2$ with respect to the integration path γ. We define $\mathcal{D} \subset \mathbb{C}$ as the open subset whose border is given by $\partial\mathcal{D} = \gamma$.

- $z_0 \notin \mathcal{D}$ and $z_0 \notin \gamma \implies \displaystyle\int_\gamma \frac{\cosh(z^2+1)}{(z-2)^3}\,dz = 0$ (by Cauchy's theorem).
- $z_0 \in \gamma \implies \displaystyle\int_\gamma \frac{\cosh(z^2+1)}{(z-2)^3}\,dz$ is ill-defined.
- $z_0 \in \mathcal{D} \implies \displaystyle\int_\gamma \frac{\cosh(z^2+1)}{(z-2)^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\cosh(z^2+1)\bigg|_{z=2} = i\pi\{2\sinh(5) + 16\cosh(5)\}$

4. (a) The point $z_0 = i$ belongs to the domain delimited by γ. We use the extended Cauchy's integral formula and recall that the integration is taken clockwise (while Cauchy's formula requires the integration to be taken counter-clockwise):

$$\int_{\gamma-} \frac{e^{iz}}{(z-i)^2}\,dz = -\int_{\gamma+} \frac{e^{iz}}{(z-i)^2}\,dz = -\frac{2\pi i}{1!}\,\frac{d}{dz}e^{iz}\bigg|_{z=i} = -2\pi i \cdot i e^{i\cdot i} = \frac{2\pi}{e}$$

(b) The point $z_0 = i$ is outside of the rectangular subset defined by $\gamma' = \{z : |\mathrm{Re}(z)| \leqslant 4,\ |\mathrm{Im}(z)| \leqslant \frac{1}{2}\}$. Cauchy's theorem indicates that this integral is zero.

(c) The path $\Gamma = \gamma_1 \cup \gamma_2$ has a different shape than the one in point (a), however the function f is holomorphic in the region between Γ and the curve γ_+ from part (a) (since f is holomorphic on $\mathbb{C} \setminus \{i\}$). Moreover, the curves Γ and γ_+ have both the same orientation and are simple closed curves. The result of the integral over Γ is therefore the same as the integral over γ_+, or, equivalently, the opposite to the final answer in (a):

$$\int_\Gamma f(z)\,dz = -\frac{2\pi}{e}. \tag{2}$$

We can now use this results to compute the integral over $\mathbb{R}$ (see also the figure below): As $R \to +\infty$, the part γ_1 of the curve covers the whole of the real axis (with orientation from $-\infty$ to $+\infty$, so we have:

$$\begin{aligned}\lim_{R\to\infty} \int_\Gamma f(z)\,dz = \lim_{R\to\infty} \int_{\gamma_1+\gamma_2} f(z)\,dz &= \lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz + \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz \\ &= \int_{-\infty}^{+\infty} f(t)\,dt + \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz.\end{aligned}$$

In view of (2), we thus have:

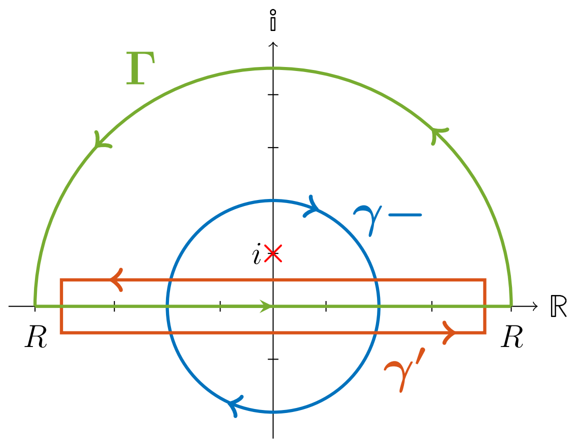
$$\int_{-\infty}^{+\infty} f(t)\,dt = -\frac{2\pi}{e} - \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz.$$

We will now parametrise γ_2 and show explicitly that $\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0$.

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^\pi f(Re^{is} \cdot Rie^{is}\,ds\right| \leqslant \int_0^\pi \left|f(Re^{is})Rie^{is}\right|\,ds = \int_0^\pi \left|\frac{e^{iRe^{is}}}{(Re^{is}-i)^2} Rie^{is}\right|\,ds \\ &\leqslant \int_0^\pi \left|\frac{R}{(Re^{is}-i)^2}\right|\,ds \overset{\star}{\leqslant} \int_0^\pi \left|\frac{R}{(R-1)^2}\right|\,ds = \frac{R\pi}{(R-1)^2}\end{aligned}$$

where the inequality with $\star$ is obtained by using the inverse triangular relation $|Re^{is} - i| \geqslant \big|\,|Re^{is}| - |i|\,\big| = |R-1|$. Therefore, this integral is necessarily zero in the limit $R \to \infty$ since its absolute value also tends to zero:

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)\,dz \right| \leqslant \lim_{R\to\infty} \frac{R\pi}{(R-1)^2} = 0$$



Paths as defined in exercise (4) with the pole at $z_0 = i$. Arrows specify the direction of rotation along the curves.

5. **(a)** The function $f(z) = e^{-z^2} = e^{-(x^2-y^2)}[\cos(2xy) + i\sin(2xy)]$ is a product of $e(\cdot)$, $\cos(\cdot)$ and $\sin(\cdot)$ which are all holomorphic on $\mathbb{C}$. Therefore, $f(z)$ is entire.

(b) Since the function is entire, its integration over any closed path $\gamma \subset \mathbb{C}$ is null (cf. Cauchy's theorem).

(c) We show that when $R \to \infty$, the integral over γ_2 also goes to zero. We first parametrise the curve: $\gamma_2(t) = R + it$ with $t \in [0, b]$ and carry the integration:

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = \lim_{R\to\infty} \int_0^b f(\gamma_2(t))\gamma_2'(t)\,dt = \lim_{R\to\infty} \int_0^b e^{-(R+it)^2} i\,dt$$

$$= \lim_{R\to\infty} e^{-R^2} \int_0^b e^{t^2} e^{-i2Rt} i\,dt \overset{\star}{=} 0$$

The equality with $\star$ derives from the fact that

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)\,dz \right| \leqslant \lim_{R\to\infty} e^{-R^2} \int_0^b \left| e^{t^2} e^{-i2Rt} i \right| dt \leqslant \lim_{R\to\infty} e^{-R^2} \underbrace{\int_0^b e^{t^2}\,dt}_{\text{finite number}} = 0$$

We obtain the same result for the computation over γ_4 since its parametrisation is equivalent.

(d) We can decompose the integration over the different parts γ_i and parametrise the curve $\gamma_3(t) = Rt + ib$ with $t \in [1, -1]$:

$$0 = \lim_{R\to\infty} \int_\gamma f(z)\,dz = \lim_{R\to\infty} \left\{ \int_{\gamma_1} f(z)\,dz + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\to 0} + \int_{\gamma_3} f(z)\,dz + \underbrace{\int_{\gamma_4} f(z)\,dz}_{\to 0} \right\}$$

$$\iff \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\,dx + \lim_{R\to\infty} \int_{1}^{-1} e^{-(R^2t^2 - b^2 + 2Rbti)} R\,dt = 0$$

$$\iff \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\,dx - \lim_{R\to\infty} \int_{-1}^{1} e^{-(R^2t^2 - b^2 + 2Rbti)} R\,dt = 0$$

$$\iff \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\sqrt{\pi}} - e^{b^2} \lim_{R\to\infty} \int_{-1}^{1} Re^{-R^2t^2} [\cos(2Rbt) + i\sin(2Rbt)]\,dt = 0$$

$$\overset{\star}{\iff} \sqrt{\pi} - e^{b^2} \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2} [\cos(2bx) + i\sin(2bx)]\,dx = 0$$

$$\iff \int_{-\infty}^{\infty} e^{-x^2} \cos(2bx)\,dx + i \int_{-\infty}^{\infty} e^{-x^2} \sin(2bx)\,dx = \sqrt{\pi}e^{-b^2} + i \cdot 0$$

By comparing the real and imaginary part of the last equivalence, we can then infer on the two relations given in this exercise. The equivalence marked by $\star$ follows from a change in variable $\{Rt \to x\ ;\ Rdt \to dx\}$.

6. If an antiderivative $F : F'(z) = f(z)$ exists, it follows that:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}[F(\gamma(t))]\,dt = F(\gamma(b)) - F(\gamma(a))$$

In particular, if we integrate f over *any* closed loop, the above equality tells us that the integral should be zero, since the function evaluated at the start and end point takes the same value.

Now consider the function $f(z) = 1/z$ integrated around the origin over $\gamma(\theta) = Re^{i\theta}$ with $\theta \in [0, 2\pi]$:

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta}\,d\theta = 2\pi i \neq 0.$$

If an antiderivative of $f(z) = 1/z$ existed on $\mathbb{C}\setminus\{0\}$, then its integral over a **closed** curve should be zero since $\gamma(0) = \gamma(2\pi)$. However, this is not the case as shown in the counter-example above.

In the domain $\mathbb{C} \setminus \left\{z : \operatorname{Re}(z) \leqslant 0, \operatorname{Im}(z) = 0\right\}$, an antiderivative of $\frac{1}{z}$ is given by $\log(z)$ (and any other antiderivative in that domain should differ from $\log(z)$ by a constant). However, $\log(z)$ cannot be continuously extended to the whole of $\mathbb{C} \setminus \{0\}$.

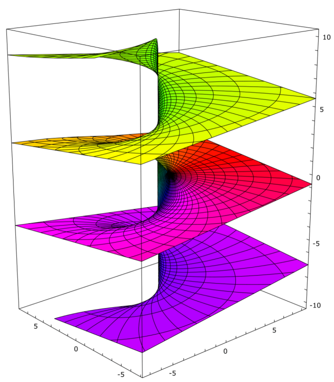


Figure 1: Representation of the multivalued argument function of $\log(z)$ around the origin. After n complete loops, the argument takes up a value of $n2\pi i$. A branch cut allows to lift this indeterminacy. Taken from: *https://en.wikipedia.org/wiki/Complex_logarithm*.